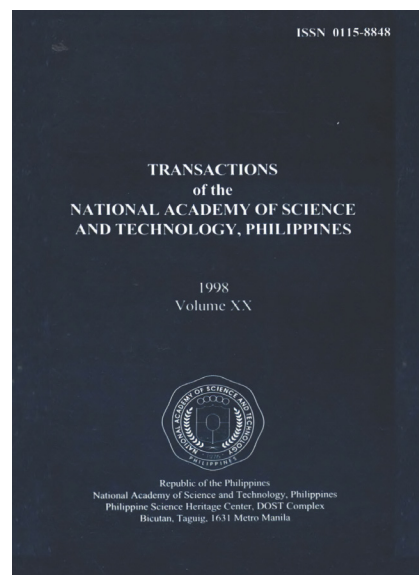


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EFFECTS OF SUBDIVISION AND CONTRACTION OF EDGES ON THE DIMENSION OF A GRAPH

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ABSTRACT

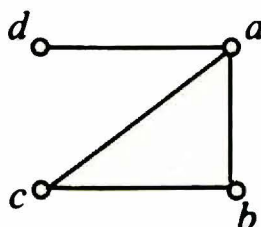
If the vertices of a graph can be associated bijectively with points in the n -dimensional Euclidean space E_n such that the distance between points associated with adjacent vertices is unity, then the graph is called a unit graph in E_n . The smallest n for which a graph G is a unit graph in E_n is called the dimension of G . Harary, et al, sometime in the 60's determined the dimension of some graphs and gave upper bounds for the dimension of a graph in terms of the number of vertices and in terms of the chromatic number. The effects of two graph operations on the dimension of a graph are considered here. An edge subdivision means inserting one new vertex in an edge of a graph. An edge contraction means reducing an edge to a single vertex by identifying its end vertices. Here, we show that the edge subdivision or edge contraction may either increase, decrease or leave the dimension of a graph unchanged. We prove here that every graph with n vertices and m edges can be subjected to a finite number of edge subdivisions to obtain a unit graph in E_2 with $n+m$ vertices and $2m$ edges. Likewise, a Hamiltonian graph with n vertices and m edges can be subjected to a finite number of edge subdivisions to yield a unit graph in E_2 with m vertices and $2m - n$ edges. Most results are proven by actual construction.

Key words: Euclidian space, distance, dimension, graph, edge subdivision, edge contraction, Hamiltonian

1. INTRODUCTION

By a *graph* we shall understand a finite, loopless graph without multiple edges. If G is a graph, we shall denote by $V(G)$ the set of vertices of G , and by $E(G)$ the set of its edges. We shall write $G = \langle V(G), E(G) \rangle$. An edge joining x and y shall be denoted by $[x,y]$.

Example. Let G be the graph defined by $V(G) = \{a, b, c, d\}$, and $E(G) = \{[a, b], [b, c], [c, a], [a, d]\}$. We represent G pictorially as follows:

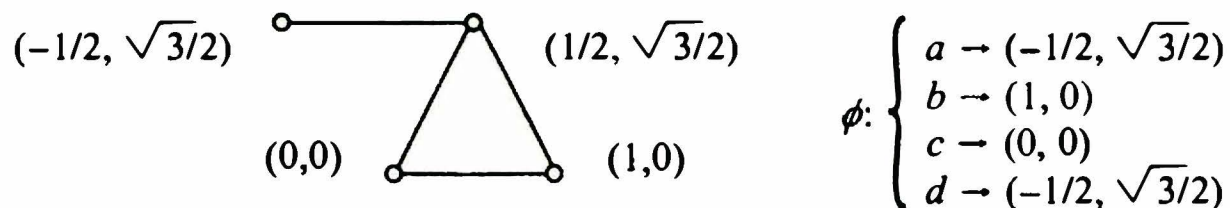


The total number of vertices in a graph is called its *order*. The *size* of a graph is the total number of edges in it. One graph which is of importance in this study is the *complete graph of order n* , denoted by K_n . The readers may please refer to (4) for other terms and concepts whose definitions are not given here.

Let n be a positive integer. The set of all ordered n -tuples (x_1, x_2, \dots, x_n) of real numbers x_i will be denoted by E_n . The elements of E_n will be called *points*. If $p = (x_1, x_2, \dots, x_n)$ and $q = (y_1, y_2, \dots, y_n)$ are two points in E_n , we define their sum as $p + q = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$. If c is any real number, we define $cp = (cx_1, cx_2, \dots, cx_n)$. Under these operations, E_n is a *vector space of dimension n* . Thus, we could also call the elements of E_n as vectors instead of points. We further define the *distance* between p and q by $d(p, q) = \{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2\}^{1/2}$. We shall refer to E_n as the *n -dimensional Euclidean space*, or the *Euclidean n -space*. For convenience, the 0-dimensional Euclidean space E_0 will be understood to be the trivial vector space, i.e., the vector space containing only the zero vector.

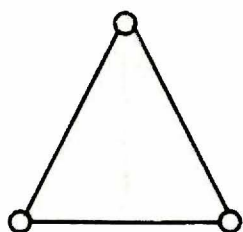
Definition 1 Let G be a graph with vertices u_1, u_2, \dots, u_n . If for some $k \geq 0$, there is one-to-one mapping $f: V(G) \rightarrow E_k$ such that the distance between $f(u_i)$ and $f(u_j)$ is 1 whenever u_i and u_j are adjacent, then we shall call f a *unit representation* of G in E_k . We shall call G a *unit graph* in E_k if G has a unit representation in E_k . The smallest k for which G is a unit graph in E_k is called the *dimension* of G , written as $\dim G$.

Example. The graph in the last example is a unit graph in E_2 and a unit representation of G in E_2 is shown below.



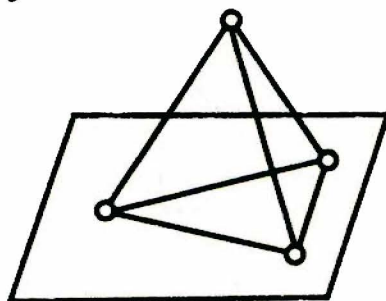
It is quite obvious that the graph in the above example has no unit representation in E_1 . Thus, it has dimension 2.

One graph of special importance is the complete graph of order n . This graph consists of n vertices which are pairwise adjacent. For example, K_3 can be described as the graph one of whose pictorial representations looks like a triangle.

A unit representation of K_3 in E_2 .

It is quite obvious that K_3 which has a unit representation in E_2 has no unit representation in E_1 . Thus, $\dim K_3 = 2$.

Forming a unit representation of K_n may be described as follows: Given $\binom{n}{2}$ sticks, each one unit long, join the sticks at their ends to produce the maximum number $\binom{n}{3}$ of congruent equilateral triangles. The solution in the case of three sticks is shown in the preceding figure. For $n = 4$, we are given $\binom{4}{2} = 6$ sticks. Therefore, we need to add three more sticks in the triangle in the last figure to form a total of $\binom{4}{3} = 4$ congruent equilateral triangles. It is easy to see that this has no solution in the plane. Thus, we are forced to go to a higher dimension. A unit representation of K_4 in E_3 is shown in the figure below.

A representation of K_4 in E_3 .

One basic question is whether every graph is a unit graph in some Euclidean n -space. This is answered by the corollary to the following theorem.

2. KNOWN RESULTS AND PRELIMINARY CONCEPTS

Theorem 1 (1), (2), (3) If K_n is the complete graph of order $n \geq 1$, then $\dim K_n = n - 1$.

Since every graph of order n is a spanning subgraph of K_n , the following corollary immediately following corollary immediately follows:

Corollary 1 If G is any graph of order n , then $\dim G \geq n - 1$.

Theorem 2 (1), (2), (3)) The complete bipartite graph $K_{m,n}$ has dimension given by the following:

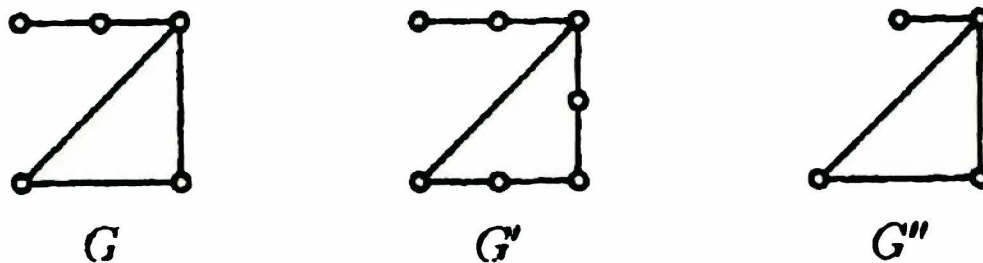
$$\dim K_{m,n} = \begin{cases} 1 & \text{if } m = 1 \text{ and } n = 1 \text{ or } 2 \\ 2 & \text{if } m = 1 \text{ and } n \geq 3 \\ 2 & \text{if } m = 2 \text{ and } n = 2 \\ 3 & \text{if } m = 2 \text{ and } n \geq 3 \\ 4 & \text{if } m \geq 3 \text{ and } n \geq 3 \end{cases}$$

Let us now define two operations on graphs, namely *subdivision* and *contraction*.

Definition 2 Let G be a graph. To *subdivide* an edge $[x, y]$ of G means to remove the edge $[x, y]$ and add a new vertex z and two new edges $[x, z]$ and $[z, y]$. A *subdivision* of G is any graph obtained from G by a finite sequence of edge subdivisions.

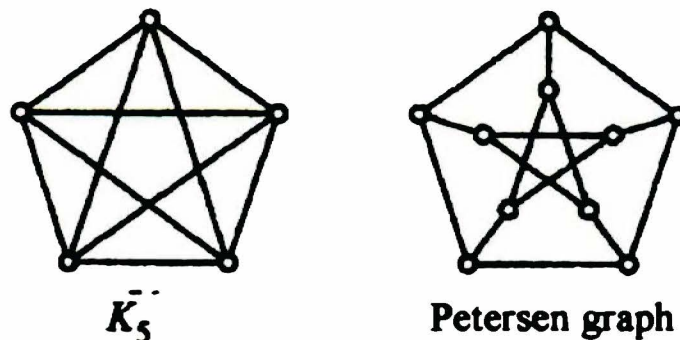
Definition 3 Let G be a graph and let $[x, y]$ be an edge of G such that x and y are not adjacent to a common vertex. To *contract* the edge $[x, y]$ means to remove the edge $[x, y]$ and to identify x and y . A *contraction* of G is any graph obtained from G by a finite sequence of edge contractions.

Example. In the figure below, G' is a subdivision of G while G'' is a contraction of G .



Definition 4 If H is a graph obtained from G by applying a finite sequence of operations consisting of subdivisions and contractions, we call H a *home-morph* of G . Two graphs G_1 and G_2 are said to be *homeomorphic* if they have respective homeomorphs H_1 and H_2 which are isomorphic.

Example. The complete graph K_5 and the Petersen graph shown below are homeomorphic.



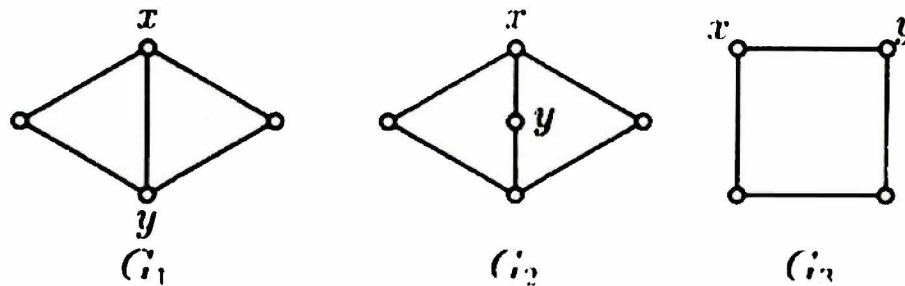
Note that a sequence of 5 contractions will transform the Petersen graph into the complete graph K_5 . No edge of K_5 may be contracted and hence it is a full contraction of the Petersen graph. This is the case where full contraction increases the dimension of the graph. The Petersen graph is of dimension 2 while the complete graph K_5 has dimension 4.

3. MAIN RESULTS

Our next two theorems give the general effects of edge subdivision and edge contraction on the dimension of a graph.

Theorem 3 *An edge subdivision may either increase, decrease, or not change the dimension of a graph.*

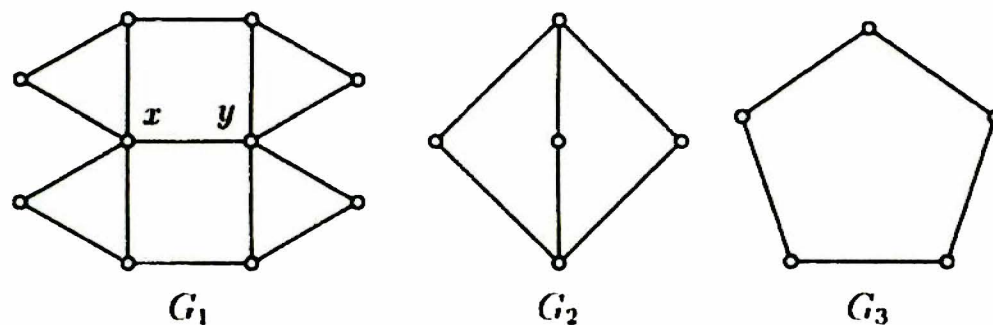
Proof: Consider the graphs G_1 , G_2 , and G_3 shown below.



It is easy to see that $\dim G_1 = 2$, $\dim G_2 = 3$ and $\dim G_3 = 2$. By subdividing the edge $[x, y]$ in each graph, we obtain graphs G'_1 , G'_2 and G'_3 . It is easy to check that $\dim G'_1 = 3$, $\dim G'_2 = 2$ and $\dim G'_3 = 2$. Thus, in the first case, there is an increase in dimension. In the second case, there is a decrease in dimension. In the last case, there is no change in dimension. \square

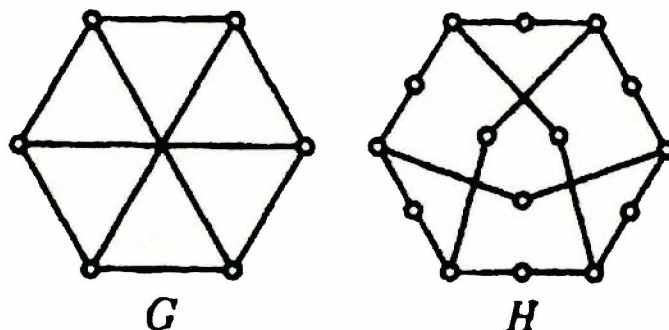
Theorem 4 *An edge contraction may either increase, decrease or not change the dimension of a graph.*

Proof: Consider the graphs G_1 , G_2 , and G_3 shown below.



For the graph G_1 , we have $\dim G_1 = 2$. By contracting the edge $[x, y]$, we get a graph whose dimension is 3. If any edge of G_2 is contracted, the dimension will change from 3 to 2. If any edge of G_3 is contracted, the dimension remains constant at 2.

Suppose each edge of a graph is subdivided, what happens to the dimension? Below, we show a graph G and the graph H obtained from G by subdividing every edge of G .

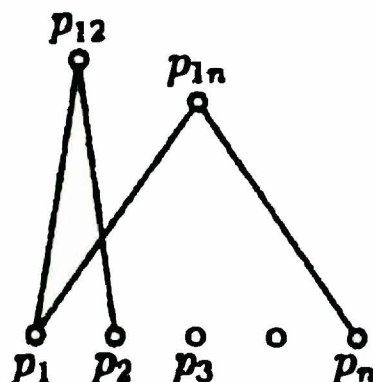


The graph G is seen to be $K_{3,3}$ and has dimension 4. We shall see later that the graph H , which is a full subdivision of G , is of dimension 2.

If we subdivide all the edges of a given graph, the result is a bipartite graph. In view of Theorem 2, the dimension of this bipartite graph is at most 4. The next theorem gives a better estimate of the dimension of the full subdivision graph of a graph.

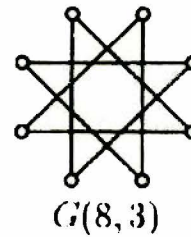
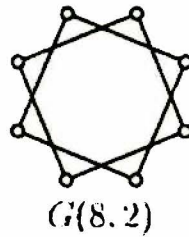
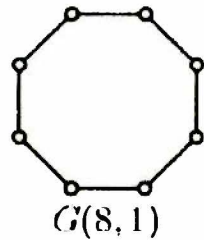
Theorem 5 *Let G be a graph of order n and size m . There exists a subdivision graph H of G of order $n+m$ and size $2m$ such that H is a unit graph in E_2 .*

Proof: Let u_1, u_2, \dots, u_n be the vertices of G and associate them with collinear points p_1, p_2, \dots, p_n in E_2 such that the distance between p_i and p_{i+1} is less than $1/n$. (Please refer to the figure below.) Then p_i and p_n are farthest apart with a distance of less than $(n-1)/n < 1$ from each other. Whenever u_i and u_j are adjacent, we introduce a new point p_{ij} in E_2 which is 1 unit away from both p_i and p_j . The total number of new points we have to add is clearly equal to m . Corresponding to p_{ij} , we subdivide the edge $[u_i, u_j]$ and introduce the subdivision vertex u_{ij} . The resulting subdivision graph of G is clearly of order $n+m$ and has a unit representation in E_2 .



Definition 5 Let $n \geq 3$ and let $1 \leq k < n/2$. We define the graph $G(n, k)$ to be the graph with vertices $1, 2, \dots, n$ whose edges are $[i, i+k]$, where $k = 1, 2, \dots, n$. The sum $i+k$ is to be read modulo n .

Some examples of graphs $G(n, k)$ are given below.



Lemma 1 Let $n \geq 3$ and let $1 \leq k < n/2$. Then $G(n, k)$ is a cycle if and only if n and k are relatively prime.

Proof: First, assume that n and k are relatively prime. We claim that the sequence $[1, 1 + k, 1 + 2k, \dots, 1 + (n - 1)k]$ is a path in $G(n, k)$. The elements of the sequence are to be read modulo n . It is clear that consecutive vertices in the sequence are edges of G , by definition. Suppose that two vertices in the path are equal, say $1 + ik \equiv 1 + jk \pmod{n}$. Then $ik \equiv jk \pmod{n}$, and $i \equiv j \pmod{n}$ since n and k are relatively prime. But $0 \leq i \leq n - 1$ and $0 \leq j \leq n - 1$. It follows that $i = j$. Now, the last vertex of the path is adjacent to the first vertex 1 because $1 + (n - 1)k + k = 1 + nk \equiv 1 \pmod{n}$. Therefore, $G(n, k)$ is a cycle.

Next, let us assume that n and k are not relatively prime and let their greatest common divisor be equal to $d > 1$. Let $n' = n/d$ and consider the sequence $[1, 1 + k, 1 + 2k, \dots, 1 + (n' - 1)k]$. Consecutive vertices of this sequence are adjacent by definition of G . If $1 + ik \equiv 1 + jk \pmod{n}$, we would have $ik \equiv jk \pmod{n}$. If we divide the congruence through by k , we get $i \equiv j \pmod{n'}$. But each of i and j ranges from 0 to $n' - 1$ only. It follows that $i = j$. The last vertex of the path is adjacent to the first vertex 1 since $1 + (n' - 1)k + k = 1 + n'k = 1 + n \left(\frac{k}{d}\right) \equiv 1 \pmod{n}$. Therefore, $G(n, k)$ contains as a subgraph a cycle of length n' which is less than n . Consequently, $G(n, k)$ is not a cycle. \square

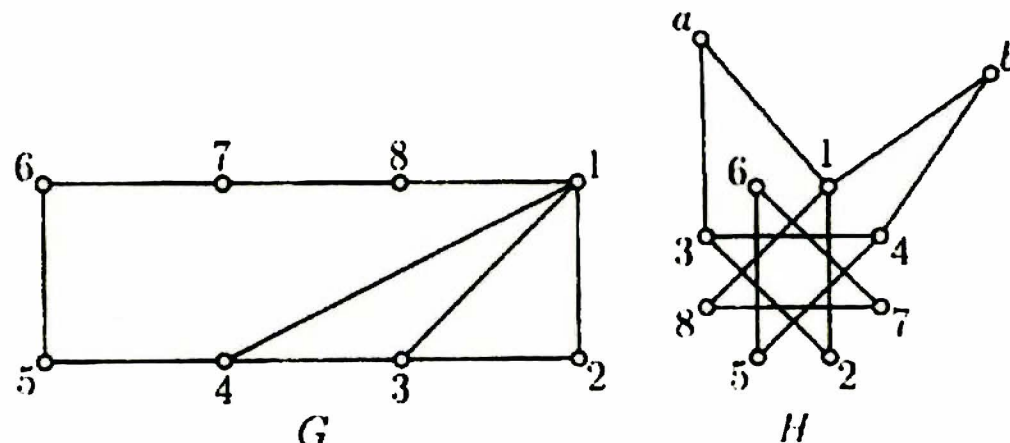
Theorem 6 Let G be a Hamiltonian graph of order $n \neq 4$ or 6, and size m . Then G has a subdivision graph H of order m and size $2m - n$ such that H is a unit graph in E_2 .

Proof: Let G be a Hamiltonian graph of order $n \neq 4$ or 6. Let us first consider the case $n = 5$. Consider a regular pentagon in E_2 whose sides have length equal to 1 unit. Let $p_1, p_2, p_3, p_4,$ and p_5 be consecutive vertices of the polygon. If $C = [u_1, u_2, u_3, u_4, u_5, u_1]$ is a Hamiltonian cycle in G , let the vertices u_1, u_2, u_3, u_4, u_5 correspond to the consecutive vertices of the Hamiltonian cycle C which are adjacent, then subdivide the edge $[u_i, u_j]$ using a vertex u_{ij} . Let u_{ij} be associated with a point in E_2 which is 1 unit away from both p_i and p_j . Clearly, such a point, say p_{ij} (not a vertex of the pentagon) is uniquely found in E_2 . Repeat this process

for every pair of adjacent but non-consecutive vertices of C . The result is a subdivision of G which is of order m and size $2m - 5$ which has a unit representation in E_2 . \square

Let G be a Hamiltonian graph of order $n > 6$. If n is even, say $n = 2^r m$, where m is odd, choose $k = 2^{r-1} m - 1$. Then n and k are relatively prime. Draw the graph $G(n, k)$ such that the edges are 1 unit long. If $[x, y]$ is an edge of G such that x and y are not adjacent in $G(n, k)$, we subdivide $[x, y]$ into two edges $[x, z]$ and $[z, y]$, each one unit long. Do this for all other similar edges. The result is a subdivision graph of H of G which is a unit graph in E_2 . Furthermore, the order of H is $n + m - n = m$ and its size is $n + 2(m - n) = 2m - n$.

Example. We illustrate in the figure below how to subdivide a Hamiltonian graph of order 8 so that the resulting graph has a unit representation in E_2 .



In the above figure, we are given a Hamiltonian graph G with spanning cycle $[1,2,3,4,5,6,7,8]$. The new vertices a and b are introduced to subdivide the edges $[1,3]$ and $[1,4]$ respectively. The graph H is the result of the edge subdivisions and a unit representation of H is shown in the same figure.

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