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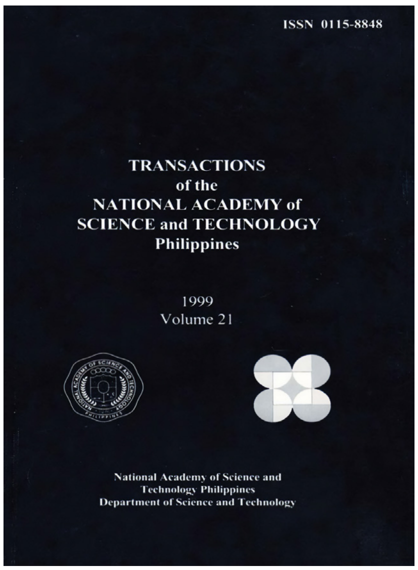

On Hierarchical Circulant Networks

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ON HIERARCHICAL CIRCULANT NETWORKS

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ABSTRACT

A fundamental consideration in the design of massively parallel and distributed computer systems is the topology of the processors (or the vertices, in graph theory). Well accepted designs are those that can be recursively decomposed, providing a way for the implementation of recursive algorithms. Hierarchical networks are recursively decomposable. In this paper, we examined the conditions for a network to be hierarchical. We limited our study to undirected and connected circulant networks.

Keywords: *Circulant networks; hierarchical; decomposition; Recursively decomposed; graph isomorphism; subgraph; edge-preserving; connected; undirected.*

1. INTRODUCTION

A fundamental consideration in the design of massively parallel and distributed computer systems is the topology of the processors or vertices. A successful topological structure must have an efficient communication scheme and programming paradigms that facilitate the design of algorithms [3].

Today, a large number of parallel machines are based on the hypercube or the binary cube interconnection network or topology because it admits optimal information dissemination schemes as well as accepts divide and conquer algorithms easily.

A number of other architectures are also well-accepted especially those that can be recursively decomposed, providing a way for the implementation of recursive algorithms. These network architectures are called hierarchical Cayley graphs. These include the k-ary n-cube, n-star and the n-pancake networks. These networks have been well-studied in the literature of parallel computer systems. According to Akers and Krishnamurthy (1984) hierarchical Cayley graphs are maximally fault tolerant. This means that the network is still connected even if one processories removed from the total number of processors adjacent to a processor.

In this paper we examined another type of hierarchical Cayley graph. This is called the hierarchical circulant networks. As of the present, only a few studies have been made on this type of hierarchical Cayley graph. In 1976, J.P. Hayes introduced the *ring-connected networks*, but if we take a closer look at the definition of *recursive circulants* by J. -H. Park and K. -Y. Chwa (1994), the ring-connected networks is a subclass of the recursive circulants. Another class of hierarchical circulant networks is the *recursive Paley graph* which was studied by this author in 1994 and formally introduced in 1996.

2. ON CIRCULAR NETWORKS

A *graph* is defined to be a pair (V, E) where V is the set of vertices and E the set of edges joining a pair of distinct vertices. A graph is *simple* and *undirected* if a pair of distinct vertices is joined by at most one edge and no edge joins the same vertex to itself. If two vertices υ_1 and υ_2 are adjacent then the symbol $\upsilon_1\upsilon_2$ (or $\upsilon_2\upsilon_1$) denotes the edge joining the two vertices. In this paper, we studied a class of graph called the circulant network.

Circulant network, (circulant graph or loop network) is an interesting network design which has attracted a number of research in interconnection networks.

Definition 1. *Undirected circulant network*

An *undirected circulant network* G of order n and degree $2t$ consists of vertices labeled from 0 to $n - 1$ such that each vertex v is adjacent to the vertices

$$\upsilon \pm s_1,\ \upsilon \pm s_2,\ \ldots, \upsilon \pm s_t$$

where the number $s_1 < s_2 < \ldots <$ st are distinct nonzero elements of Z_n and together with their negative counterparts are called the jump sizes of G. We denote this circulant network by $G(n; s_1, s_2, \ldots, s_t)$.

Example 1. *The circulant network* $G(24; 1, 2, 4)$

The circular network $G(24; 1, 2, 4)$ has 6 jump sizes: namely, ± 1, ± 2, and ± 4. Vertex 3 is adjacent to 6 vertices. These are:

1. 4, since $4 \equiv 3 + 1 \pmod{12}$;
2. 5, since $5 \equiv 3 + 2 \pmod{12}$;
3. 7, since $7 \equiv 3 + 4 \pmod{12}$;
4. 2, since $2 \equiv 3 - 1 \pmod{12}$;
5. 1, since $1 \equiv 3 - 2 \pmod{12}$; and
6. 11, since $-1 + 3 - 4 \pmod{12}$. Note that under *addition modulo* 12, the numbers -1 and 11 are equivalent.

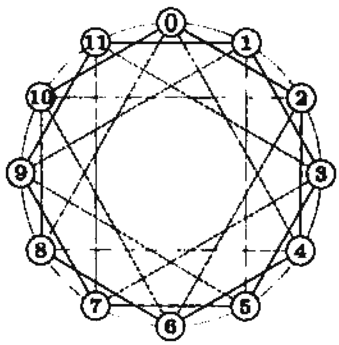


Figure 1. The Circulant Network $G(24; 1, 2, 4)$

3. ON SUBGRAPHS AND GRAPH ISOMORPHISM

Let us examine some of the properties of the graph which are important in this paper. In particular, let us define a subgraph of a graph, graph isomorphism, and hierarchical networks.

Definition 2. *Subgraph of a Graph*

A graph H is a *subgraph of a graph* G if the vertices of H are in G and the edges of H are in G.

Example 2. *A subgraph of* $G(24; 1, 2, 4)$.

The vertices 0, 2, 4, 6, 8, and 10 induced a subgraph of $G(24; 1, 2, 4)$ which we shall denote by $\Gamma(\langle 2\rangle)$. The edges of this new graph are in $G(24; 1, 2, 4)$ where the jump sizes involved are ± 2 and ± 4.

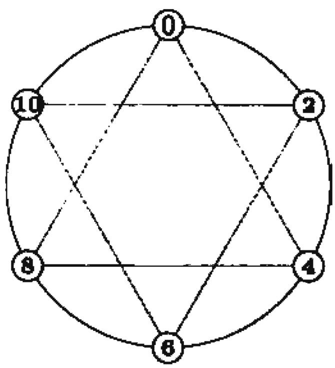


Figure 2. The subgraph $\Gamma(\langle 2\rangle)$.

Definition 3. *Graph Isomorphism*

A graph G is isomorphic to another graph H if there exists an isomorphism ϕ from t he vertex set of G into the vertex set of H such t hat $\upsilon_1\upsilon_2$ is an edge in G if and only if $\phi(\upsilon_1)$ $\phi(\upsilon_2)$ is an edge in H. In order to show t hat two graphs are isomorphic, we have to find a one-to-one function and prove that this function preserves the edges (edge-preserving).

Example 3. *The graph* $G(6; 1, 2)$ *and* $\Gamma(\langle 2 \rangle)$

Consider the graph $G(6; 1, 2)$ of order 6 and jump sizes ± 1 and ± 2 and the graph $\Gamma(\langle 2 \rangle)$ as defined in the previous illustration.

The isomorphism ϕ: $V(G) \rightarrow V(H)$ where $V(G)$ is the vertex set of $G(6; 1, 2)$ and $V(H)$ is the vertex set of $\Gamma(\langle 2 \rangle)$, is defined by

$$\phi(\upsilon) \equiv 2 \cdot \upsilon \text{ (mod 1)2} \qquad (1)$$

where the operation "•" is the product taken under module 12.

The function is well-defined and a bijection.

- ϕ is one-to-one:

  $\phi(\upsilon_1) \equiv \phi(\upsilon_2)$ (mod 12)

  $2 \cdot \upsilon_1 \equiv 2 \cdot \upsilon_2$ (mod 12)

  [redacted] υ_2 (mod 6)

  Since υ_1, $V_2 \in V(G)$, i.e. $0 \leq \upsilon_1, \upsilon_2 \leq 5$, it follows that $\upsilon_1 = \upsilon_2$. Hence, ϕ is one-to-one.

υ	$\phi(\upsilon)$
0	0
1	2
2	4
3	6
4	8
5	10

Table 1: Image under ϕ

- ϕ is onto:

  Let h be a vertex in $\Gamma(\langle 2 \rangle)$. Then h is one of 0, 2, 4, 6, 8, 10. See Table 2.1. Hence, we can write $h = 2.h'$ where $h' \in \{0, 2, 3, 4, 5\}$. Take h'. Then $\phi(h') = 2 \cdot h'$ (mod 12). Hence, ϕ is onto.

- ϕ is edge-preserving:

  Let $\upsilon_1, \upsilon_2 \in V(G)$.

  $\upsilon_1 \upsilon_2 \in E(G)$ if and only $\upsilon_2 \equiv \upsilon_1 + s$ (mod 6) where $s = 1$ or $s = 2$. Thus, $\upsilon_2 = \upsilon + s + 6x$ for some integer x.

$$\begin{aligned}\phi(\upsilon_2) &\equiv 2\cdot\upsilon_2 \pmod{12}\\ &\equiv 2\cdot(\upsilon_1 + s + 6x) \pmod{12}\\ &\equiv 2\cdot\upsilon_1 + 2.s + 2\cdot 6x \pmod{12}\\ &\equiv 2\cdot\upsilon_1 + 2s + 12x \pmod{12}\\ &\equiv \phi(\upsilon_1) + 2s \pmod{12}\end{aligned}$$

Since 2s = 2 or 2s = 4, it follows that $\phi(\upsilon_1)\phi(\upsilon_2) \in E(H)$.
Now, if $\phi(\upsilon_1)\phi(\upsilon_2) \in E(H)$, then for s = 1 of s = 2,

$$\begin{aligned}\phi(\upsilon_2) &\equiv \phi(\upsilon_1) + 2s \pmod{12}\\ 2\cdot\upsilon_2 &\equiv 2\cdot\upsilon_1 + 2s \pmod{12}\\ \upsilon_2 &\equiv \upsilon_1 + s \bmod 6\end{aligned}$$

Since s = 1 or s = 2, it follows that $\upsilon_1\upsilon_2 \in E(G)$.
Hence, ϕ is edge-preserving.
Therefore, $G(6; 1, 2) \cong \Gamma(\langle 2\rangle)$.

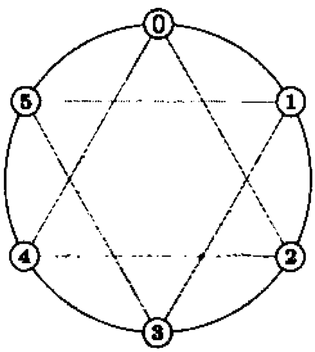


Figure 3(a): The graph G(6; 1, 2)

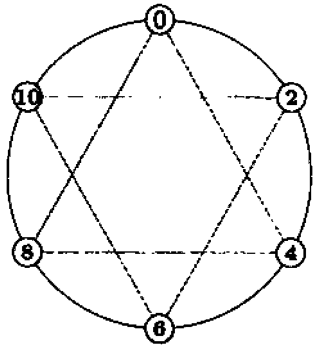


Figure 3(b): The graph $\Gamma(\langle 2\rangle)$

4. HIERARCHICAL CAYLEY GRAPH

Let *G* be a finite group with a set of generators *S*.

Definition 4. *Cayley Graph*

The Cayley Graph of *G* with respect to the generating set *S* is the graph with vertex set *G*, where *g* and *gs* are adjacent, for g ∈ *G* and *s* ∈ *S*.

Lemma 1 *If S is closed under inverses then G is undirected or bidirectional.*

Proof. This is straightforward. If node a is adjacent to node b, where the direction is from a to b, then there exists $s \in S$ such that $b = as$. But, $bs^{-1} = a$. Since S is closed under inverses, $s^{-1} \in S$. Hence, node b is adjacent to node a. This gives the direction from b to a.

Therefore, the lemma follows.

Cayley graphs are popular in the design of parallel and distributed systems because of some of their properties, like vertex transitivity.

A circulant network is Cayley if the set of hops generate all the nodes of the circulant network. The configuration $G(12; 2, 4)$ is a circulant network but not Cayley since node 1 cannot be expressed as a linear combination of 2 and 4 under modulo 12. Also, this graph is not connected. See Fig. 4.

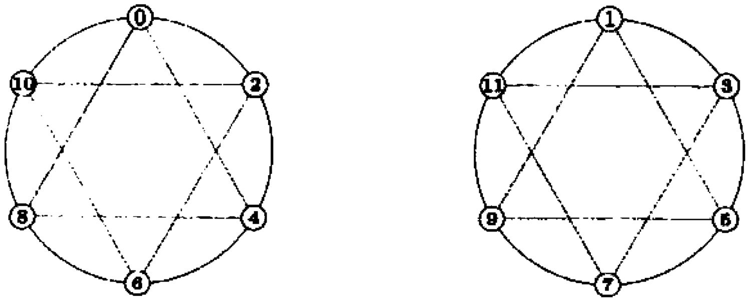


Figure 4. Circulant network $G(12; 2, 4)$

Let us have the following lemma to show the necessary and sufficient condition for a circulant network to be connected.

Lemma 2. *A circulant network is connected if and only if the greatest common divisor of* $a_1, a_2, \ldots, a_t$ *and* N *is equal to 1.*

Proof. Let us denote the greatest common divisor of a$a_1, a_2, \ldots, a_t$ and N by $gcd(a_1, a_2, \ldots, a_t, N)$.

Suppose that $gcd(a_1, a_2, \ldots, a_t, N) = 1$.

Then there exist integers $q_1, q_2, \ldots, q_t, r$ such that

$$a_1 q_1 + a_2 q_2 + \ldots + a_t q_t + Nr = 1.$$

This implies that $a_1 q_1 + a_2 q_2 + \ldots + a_t q_t + Nr \equiv 1 \pmod N$. Thus, node 1 and consequently all nodes can be reached from vertex 0. Hence, the circulant network is connected.

Suppose that $gcd(a_1, a_2, \ldots, a_t, N) = d > 1$. Then only nodes that are multiples of d are reachable from 0.

Therefore, the circulant network is not connected.

The following theorem that a connected circulant network is a Cayley graph.

Theorem 1. *A connected circulant network is Cayley.*

Proof. This is immediate from the previous lemma, since a node of a connected circulant network can be expressed as a linear combination of the different hops of the circulant network

Another property of a graph being considered is recursive decomposition which provides a way for the implementation of recursive algorithms (Berthome et al.) Hierarchical Cayley graphs can be recursively decomposed.

Definition 5. *Hierarchical Cayley Graph (Berthome et al.)*

A Cayley graph G is said to be *hierarchical* if it can be decomposed into a collection $\kappa(n)$ isomorphic subgraphs along with edges connecting them, where $\kappa(n) < |G|$. Each subgraph is a smaller Cayley graph from the same family as the original graph.

Among the hierarchical Cayley graphs are the k-ary n-cubes including the hypercubes, n-star graphs, recursive circulants and recursive Paley graphs.

Example 4. *The circulant network $G(12;\ 1,\ 2,\ 4)$ is hierarchical since we can decomposed it into a collection of 2 isomorphic subgraphs.*

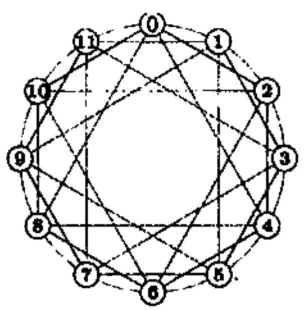
The subgraphs are:

1. $\Gamma(\langle 2 \rangle)$.
2. Induced subgraph of $\{1, 3, 5, 7, 9, 11\}$.

These subgraphs belong to the same family of $G(12;\ 1,\ 2,\ 4)$, the undirected and connected circulant networks.

Each of these subgraphs can be decomposed again into 3 subgraphs which are isomorphic to $G(3;\ 1)$. See Fig. 5.

The circulant network G(12; 1, 2, 4)



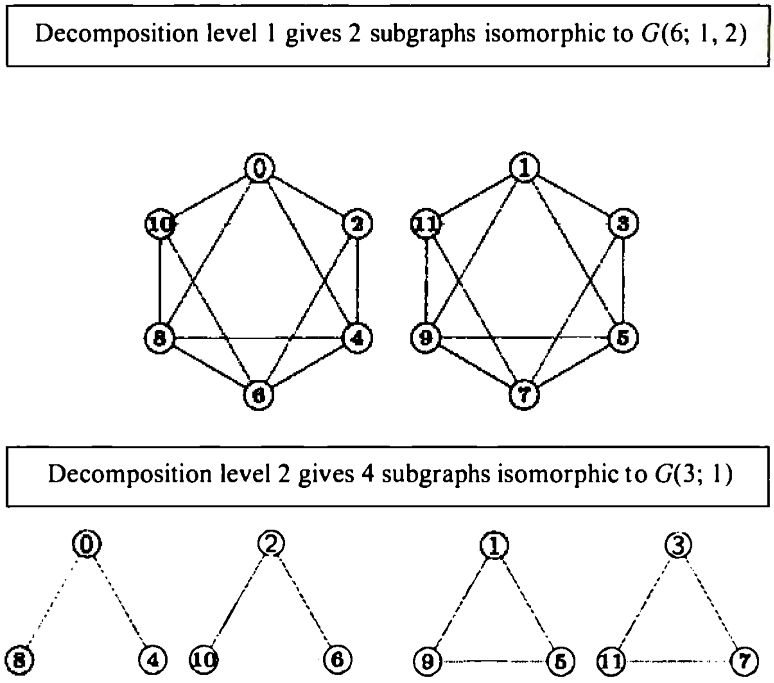


Figure 5. Recursive Decomposition of $G(12;\ 1, 2, 4)$

Definition 6. *Decomposition Level and Recursion Depth of the Hierarchical Cayley Graph*

We denote *decomposition level 1* as the first decomposition of the original graph into isomorphic Cayley subgraphs which belong to the same family of the original graph. If we can decompose these Cayley subgraph into isomorphic Cayley subgraphs belonging to the same family of the original graph, then this is *decomposition level 2.* If we can do this type of decomposition n times, then we reach *decomposition level n.* If after decomposition level *n*, we can not decompose the resulting subgraphs anymore, then the length of the decomposition is *n*.

The *recursion depth* of the hierarchical Cayley graph is the maximum length of all possible decomposition of the graph.

Example 5. *The recursion depth of* $G(12;\ 1, 2, 4)$

As shown in Fig. 5, the recursion depth of the hierarchical circulant network $G(12;\ 1, 2, 4)$ is 2.

Example 6. *The recursion depth of* $G(24; 1, 2, 3, 4, 6, 8)$

To find all the decompositions of the hierarchical circulant networks that give the maximum length, we consider all its maximal circulant subgraphs.

A *maximal circulant subgraph* of a circulant network G is a circulant subgraph of G which is not contained in any other circulant subgraph of G.

The two maximal circulant subgraphs of $G(24; 1, 2, 3, 4, 6, 8)$ are the subgraphs.

1. $\Gamma(\langle 2\rangle)$ induced by $< 2 > = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 0\}$. See Fig. 6(a); and
2. $\Gamma(\langle 3\rangle)$. induced by $< 3 > = \{3, 6, 9, 12, 15, 18, 21, 0\}$ See Fig. 6(b)

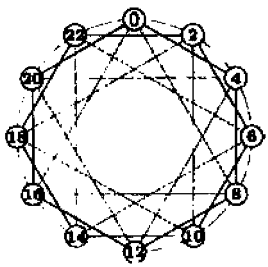


Figure 6(a). The Sugraph $\Gamma(\langle 2\rangle)$

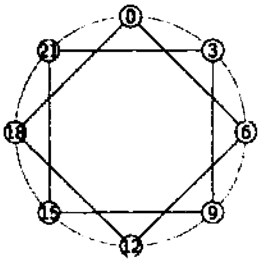


Figure 6(b). The Sugraph $\Gamma(\langle 3\rangle)$

The subgraph $\Gamma(\langle 2\rangle)$ is isomorphic to $G(12; 1, 2, 4)$ which his hierarchical of recursion depth equal to 2. Since we can decompose $G(24; 1, 2, 3, 4, 6, 8)$ into two isomorphic circulant subgraphs; namely.

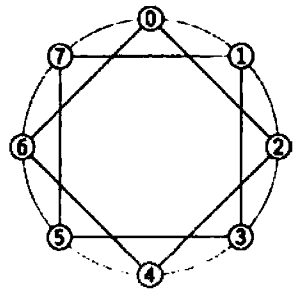
1. $\Gamma(\langle 2\rangle)$ and
2. The subgraph induced by $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23\}$, the length of this decomposition is 3.

The subgraph $\Gamma(\langle 3\rangle)$ is isomorphic to $G(8; 1, 2)$ which can be decomposed into two isomorphic subgraphs; namely, the subgraph induced by $\{0, 2, 4, 6\}$ and the subgraph induced by $\{1, 3, 5, 7\}$. This induced subgraph is isomorphic to $G(4; 1)$, which is a cycle. The decomposition stops here. The length of this decomposition is 2. See Fig. 7

Since the maximum of the two lengths of decomposition is 3, it follows that the recursion depth of $G(24; 1, 2, 3, 4, 6, 8)$ is 3.

It is the aim of this paper to find the conditions for which undirected and connected circulant networks are hierarchical. The project is motivated by Akers and Krishnamurthy (1984) who showed that hierarchical Cayley graphs are *maximally fault tolerant.*

The circulant network G(8; 1, 2)



Decomposition level 1 gives 2 subgraphs isomorphic to $G(6;\ 1.\ 2)$

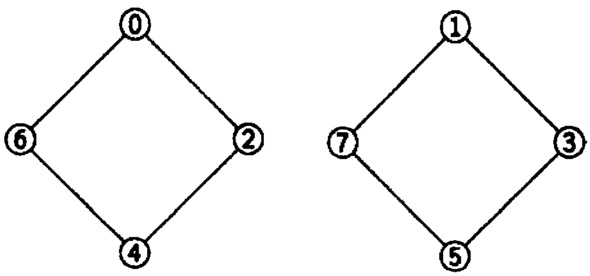


Figure 7. Recursive decomposition of G(24; 1,2,3,4,6,8)

5. CONDITIONS FOR HIERARCHY

Let us now discuss the conditions for an undirected and connected circulant graph to be hierarchical.

Theorem 2. *Let $n = ms^e$ for some positive integers m, e, s and s, $r > 1$. Then the circulant network $G(n;\ u,\ s^{t_1},\ s^{t_2}, \ldots,\ s^{t_k})$ is hierarchical of recursion depth k where $gcd(u,n) = 1$, $t_1 < t_2 < \ldots < t_k \leq e$ and $t_{j+1} - t_j = d$ for $j = 1, 2, \ldots, k-1$.*

Proof. Let $G = G(n;\ u,\ s^{t_1},\ s^{t_2}, \ldots,\ s^{t_k})$ and let $S(i,j) = \{j,\ s^{t_i},\ + j,\ 2s^{t_i},\ + j, \ldots,\ (ms^{e-t_i} - 1)\, s^{t_i} + j\}$ *or* $i = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, s^{t_1} - 1$.

If $gcd(n,u) = 1$, then $gcd(n,u,\ s^{t_1},\ s^{t_2}, \ldots,\ s^{t_k}$. Thus, G is connected.

Since $t_{j+1} - t_j = d$,

$$\begin{aligned} t_{i+r} - t_i &= (t_{i+r} - t_{i+r-1}) + (t_{i+r-1} - t_{i+r-2}) + \ldots + (t_{i+2} - t_{i+1}) + (t_{i+1} - t_i) \\ &= \underbrace{d + d + \ldots + d + d}_{r \text{ terms}} \\ &= rd \end{aligned}$$

Let us consider the largest possile value of j which is $s^{t_i} - 1$. Then the largest possible element of $S(i, j)$ where j is the maximum is

$$\begin{aligned}(ms^{e-t_i} - 1) s^{t_i} + s^{t_i} - 1 &= (ms^{e-t_i} - 1) s^{t_i} - 1 \\ &= ms^e - s^{t_i} + s^{t_i} - 1 \\ &= ms^e - 1\end{aligned}$$

Since $j < s^{t_i} + j < 2s^{t_i} + j < \ldots < (ms^e - s^{t_i} + j \leq ms^e - 1$, the addition and multiplication are taken under the entire set of integers and not under modulo n.

Claim 1: For fixed i and j, $|S(i,j)| = ms^{(k-i)d}$.

This is clear from the definition of $S(i, j)$.

Claim 2: For fixed i and j, the elements in $S(i, j)$ are distinct.

Suppose that $as^{t_i} + j = bs^{t_i} + j$. Then

$$\begin{aligned}as^{t_i} + j &= bs^{t_i} + j \\ as^{t_i} &= bs^{t_i} \\ a &= b.\end{aligned}$$

Therefore, for fixed i and j, the elements in $S(i, j)$ are distinct.

Claim 3: For each fixed i, $S(i, j_1, \cap S(i, j_2) = \emptyset$

Suppose that $\upsilon \in S(i, j_1) \cap S(i, j_2)$. Then $\upsilon \equiv as^{t_i} + j_1 \pmod{n}$ and $\upsilon \equiv bs^{t_i} + j_2 \pmod{n}$.

Without loss of generality, let us assume that $b \geq a$. Then

$$\begin{aligned}as^{t_i} + j_1 &= bs^{t_i} + j_2 \\ j_1 - j_2 &= (b - a)s^{t_i}\end{aligned}$$

Since $b > a$, it follows that $j_1 > j_2$ and $j_2 - j_1$ is an integral multiple of s^{t_i} which implies that $j_2 - j_1 > s^{t_i}$. But $0 \geq j_1, j_2 < s^{t_i}$ implies that $j_1 - j_2 < s^{t_i}$. Hence, we have a contradiction. Thefore, $\upsilon \notin S(i, j_1) \cap S(i, j_2)$ for any υ, i.e. $S(i, j_1) \cap S(i, j_2) = \emptyset$.

Claim 4: For each fixed i, $S(i,j)$ partitions the vertex set of G into s^{t_i} partitions.

For each fixed i, we have

$$\begin{aligned}\sum_{j=0}^{s^{t_i}-1} | S(i, j) | &= \sum_{j=1}^{s^{t_i}} |S(i, j)| \\ &= \sum_{j=1}^{s^{t_i}} ms^{e-t_i} \\ &= s^{t_i} . ms^{e-t_i} \\ &= ms^e \\ &= |V(G)|\end{aligned}$$

Thefore, from Claims 1, 2, 3 and 4, for each fixed i, $S(i, j)$ partitions the vertex set of G into s^{t_i} partitions where each partition contains ms^{e-t_i} elements.

Consider the subgraph induced by $S(i, j)$ where i and j are fixed. We denote this by $\Gamma(S(i, j))$.

Let $as^{t_i} + j$ and $bs^{t_i} + j$ be two vertices in $\Gamma(S(i, j))$ that are adjacent. Then for some positive integer $l \geq k$,

$$\begin{aligned} bs^{t_i} + j &\equiv (as^{t_i} + j) + s^{t_i} + j \pmod{n} \\ bs^{t_i} &\equiv as^{t_i} + s^{t_i} \pmod{n} \\ b &\equiv a + s^{t_l - t_i} \pmod{ms^{e-t_i}} \end{aligned}$$

Hence $t_l \geq t_i$. This implies that $l \geq i$. Since s^l is a jump size, it follows that $i \leq l \leq k$. Thereore, the jump sizes in $\Gamma(S(i,j))$ are $\pm s^{t_i}, \pm s^{t_i}, \ldots, \pm s^{t_k}$.

Claim 5: $G(ms^{(k-i)d}; 1, s^d, s2^d, \ldots, s^{(k-i)d}) \cong \Gamma(S(i, j))$ such that $\Gamma(S(i, j))$ is the subgraph induced by $S(i, j)$ where i and j are fixed.

Let $G(ms^{e-t_i}; 1, s^d, s2^d, \ldots, s^{(k-i)d})$.

Define a function ϕ: $V(G_i) \rightarrow V(\Gamma(S(i, j)))$ by

$$\upsilon = \upsilon s^{t_i} + j \qquad (2)$$

Since i and j are fixed, ϕ is well-defined

To show that ϕ is one-to-one:

Suppose that $\phi(\upsilon_1) = \phi(\upsilon_2)$ for $\upsilon_1, \upsilon_2 \in V(G_i)$. Then

$$\begin{aligned} \phi(\upsilon_1) &= \phi(\upsilon_2) \\ \upsilon_1 s^{t_i} + j &= \upsilon_2 s^{t_i} + j \\ \upsilon_1 s^{t_i} &= \upsilon_2 s^{t_i} \\ \upsilon_1 &= \upsilon_2 \end{aligned}$$

Hence, ϕ is one-to-one.

To show that ϕ is onto:

Let $w \in V(\Gamma(S(i, j)))$. Then $w = \upsilon s^{t_i} + j$. *Thus, take* $\upsilon \in V(G_i)$. Hence, ϕ is onto.

To show that ϕ is edge-preserving:

Let $\upsilon_1, \upsilon_2 \in V(G_i)$.

$\upsilon_1\upsilon_2 \in E(G_i) \Longleftrightarrow \upsilon_2 \equiv \upsilon_1 + s^{rd} \pmod{ms^{e-t_i}}$ where $r = 0, 1, \ldots, k - i$

$\Longleftrightarrow \upsilon_2 \equiv \upsilon_1 + s^{rd}\, xms^{e-t_i}$ for some integer x

$$
\begin{aligned}
\phi(\upsilon_2) &= \upsilon_2 s^{ti} + j \\
&= (\upsilon_2 + s^{rd} + xms^{e-ti})\, s^{ti} + j \\
&= (\upsilon_1 s^{ti} + j) + s^{rd+ti} + xms^{e-ti+ti} \\
&= \phi(\upsilon_1) + s^{ti+r-ti+ti} + xms^{e} \\
&= \phi(\upsilon_1) + s^{ti+r} + xms^{e} \\
&\equiv \phi(\upsilon_1) + s^{ti+r} \pmod{ms^{e}} \\
&\equiv \phi(\upsilon_1) + s^{ti+r} \pmod{n}
\end{aligned}
$$

Since $r = 0, 1, \ldots, k - i$, s^{ti+r} is a jump size of $\Gamma(S(i, j))$.

Hence, $\phi(\upsilon_1)\phi(\upsilon_2) \in V(\Gamma(S(i, j)))$.

Suppose that $\phi(\upsilon_1)\phi(\upsilon_2) \in E(\Gamma(S(i, j)))$.

Then for some $r = 0, 1, \ldots, k - i$,

$$
\begin{aligned}
\phi(\upsilon_1) &\equiv \phi(\upsilon_1) + s^{ti+r} \pmod{n} \\
\upsilon_2 s^{ti} + j &\equiv (\upsilon_1 s^{ti} + j) + s^{ti+r} \pmod{n} \\
\upsilon_2 s^{ti} &\equiv \upsilon_1 s^{ti} + s^{ti+r} \pmod{ms^{e}} \\
\upsilon_2 &\equiv \upsilon_1 + s^{ti+r-ti} \pmod{ms^{e-ti}} \\
\upsilon_2 &\equiv \upsilon_1 + s^{rd} \pmod{ms^{e-ti}}
\end{aligned}
$$

Since s^{rd}, for $r = 0, 1, \ldots, k - i$, is a jump size in G_i, it follows that $\upsilon_1\upsilon_2 \in E(G_i)$.

Hence, for a fixed i and j, $G_i \cong (\Gamma(S(i, j))$.

This implies also that for a fixed i, $(\Gamma(S(i, j_1)) \cong (\Gamma(S(i, j_2))$, for any pair $j_1, j_2 \in \{0, 1, \ldots, s^{ti} - 1\}$.

Thus, the circulant network $G(n;\ u, s^{t1}, s^{t2}, \ldots, s^{tk})$ can be decomposed into s^{ti} isomorphic subgraphs that are isomorphic to the circulant network $G(ms^{e-ti};\ 1, s^{d}, s2^{d}, \ldots, s^{(k-i)d})$.

Since i varies from 1 to k, the decomposition length is k. Since there are k + 1 distinct jump sizes (up to magnitude), k is the maximum, and hence, the recursion depth of $G(n;\ u, s^{t1}, s^{t2}, \ldots, s^{tk})$ is k.

Therefore, $G(n;\ u, s^{t1}, s^{t2}, \ldots, s^{tk})$ is a hierarchical circulant network of recursion depth k.

Example 7. *The circulant network* $G(192; 1, 2, 8, 32)$

$$\begin{array}{ccc} & & G(192;\ 1, 2, 8, 32) \\ & & \downarrow \\ G(96; 1, 4, 16) & \cong & \Gamma(\langle 2 \rangle). \\ & & \downarrow \\ G(24; 1, 4) & \cong & \Gamma(\langle 8 \rangle). \\ & & \downarrow \\ G(6; 1) & \cong & \Gamma(\langle 16 \rangle). \end{array}$$

where

$\Gamma(\langle 2 \rangle)$ is the induced subgraph of $\{0, 2, 4, \ldots, 190\}$

$\Gamma(\langle 8 \rangle)$ is the induced subgraph of $\{0, 8, 16, \ldots, 184\}$

$\Gamma(\langle 16 \rangle)$ is the induced subgraph of $\{0, 32, 64, 96, 128, 160\}$

Since $\Gamma(\langle 2 \rangle) \cong \Gamma(\{1, 3, 5, \ldots, 191\})$, we can decompose $G(192, 1, 2, 8, 32)$ into 2 circulant isomorphic subgraphs.

In the next level of decomposition, we have 8 isomorphic subgraphs induced by $(\{j, 8+j, 16+j, 184+j\}$ where $j = 0, 1, 2, 3, 4, 5, 6, 7$. These subgraphs are isomorphic to $G(24; 1, 4)$.

In the last level of decomposition, we have 32 isomorphic subgraphs induced by $\{j, 32+j, \ldots, 64+j, 96+j, 128+160+j\}$ where $j = 0, 1, 2, \ldots, 30, 31$.

The number of jump sizes of $G(192, 1, 2, 8, 32)$ (up to magnitude) is 4, hence 3 is the maximum of the length of all decompositions of $G(192, 1, 2, 8, 32)$.

Therefore, $G(192; 1, 2, 8, 32)$ is a hierarchical circulant network of recursion depth 3.

Let $n = ms^e$ for some positive integers m, e, s and $s, r > 1$.

Consider $G(n; u, ws^{t_1}, ws^{t_2}, \ldots, ws^{t_k})$ where $gcd(u, n) = 1$, $gcd(w, n) = 1$, $t_1 < t_2 < \ldots < t^k \leq e$ and $t_{j+1} - t_j = d$ for $j = 1, 2, \ldots, k-1$.

Theorem 3 *The circulant network* $G(ms^{e-t_i}; 1, s^d, s^{2d}, \ldots, s^{(k-i)d})$ *is isomorphic to a subgraph of G.*

Also, G is a hierarchical circulant network of recursion depth k.

Proof. Let $S(i, j) = \{j, ws^{t_i}+j, 2ws^{t_i}+j, \ldots, (ms^{e-t_i}-1)ws^{t_i}+j\}$ for $i = 1, 2, \ldots, k$ and $j = 0, 1, \ldots, s^{t_i}-1$. Using the proof in the previous theorem, we can easily show that for fixed i, the set $S(i, j)$ partitions $V(G)$ into ms^{e-t_i} subsets with equal number of elements.

Consider the induced subgraph $\Gamma S(i, j))$ for a fixed i and j. We will show that this isomorphic to the circulant network $G(ms^{e-ti}; 1, s^d, s^{2d}, \ldots, s^{(k-i)d})$ which we shall denote by G_i.

Define a function ϕ: $V(G_i) \rightarrow V(\Gamma(S(i, j)))$ by

$$\upsilon \equiv \upsilon w s^{ti} + j \pmod{n}. \tag{3}$$

Since i and j are fixed and w is constant, ϕ is well-defined.

To show that ϕ is one-to-one:

Suppose that $\phi(\upsilon_1) = \phi(\upsilon_2)$ for $\upsilon_1, \upsilon_2 \in V(G_i)$. Then

$$\begin{aligned} \phi(\upsilon_1) &= \phi(\upsilon_2) \\ \upsilon_1 w s^{ti} + j &\equiv \upsilon_2 w s^{ti} + j \pmod{n} \\ \upsilon_1 w &\equiv \upsilon_2 w \pmod{ms^{e-ti}} \\ \upsilon_1 &\equiv \upsilon_2 \pmod{ms^{e-ti}} \text{ since } gcd(w, n) = 1. \end{aligned}$$

Since $0 \leq \upsilon_1, \upsilon_2 < ms^{e-ti}$, it follows that $\upsilon_1 = \upsilon_2$.

Hence, ϕ is one-to-one.

To show that ϕ is onto:

This is similar to the proof in the previous theorem.

To show that ϕ is edge-preserving:

Again the proof is similar to that of the previous theorem.

Hence, $G(ms^{e-ti}; 1, s^d, s^{2d}, \ldots, s^{(k-i)d}) \cong \Gamma(S(i, j))$.

Thus, we can decompose G into ms^{e-ti} isomorphic circulant subgraphs. Thus, G is hierarchical.

From the pevious theorem, $G(ms^{e-ti}; 1, s^d, s^{2d}, \ldots, s^{(k-i)d})$ can be decomposed recursively $k - 1$ times.

Therefore, the recursion depth of the hierarchical network G is k.

Example 8. *The circulant network* $G(320; 1, 6, 24, 96)$.

$$\begin{array}{ccc} & & G(G(320; 1, 6, 24, 96)) \\ & & \downarrow \\ G(160, 1, 4, 16) & \cong & \Gamma(\langle 6 \rangle) \\ & & \downarrow \\ G(40, 1, 4) & \cong & \Gamma(\langle 24 \rangle) \\ & & \downarrow \\ G(10, 1,) & \cong & \Gamma(\langle 96 \rangle) \end{array}$$

Like the previous example, we can easily show that $G(320, 1, 6, 24, 96)$ is a hierarchical circulant network of recursion depth 3.

Theorem 4. *$G(n; u, s_1, s_2, \ldots, s_k)$ is a hierarchical circulant network if the following conditions are satisfied:*

1. *$gcd(n, u) = 1$; and*
2. *$gcd(n, s_i) = r_i$ and $r_i | r_1$ where i - 1, 2,..., t*

Proof. *Let $G' = G(m_1; s_1', s_2', \ldots, s_t')$ where $n = m_1 r_1$, $r_i = s_i' r_i$* for $i = 1, 2, \ldots, t$.

Consider the set $S(j) = \{j, s_1 + j, 2s_1 + j, \ldots, (m_1 - 1)s_1 + j\}$ for $j = 0, 1, 2, \ldots, r_1 - 1$.

Clearly, the elements of $S(j)$ are distinct and $S(j)$ for $j = 0, 1, 2, \ldots, r_1 - 1$ partitions $V(G)$, the vertex set of $G(n; u, s_1, s_2, \ldots, s_t)$.

Consider the induced subgraph $\Gamma(S(i, j))$. We show that $G' \cong \Gamma(S(i, j))$.

For a fixed j, define a function ϕ: $V(G') \rightarrow V(\Gamma(S(i, j)))$ by

$$\upsilon \equiv \upsilon c_1 r_1 + j \pmod{n}. \quad (4)$$

where $c_1 r_1 = s_1$ and $gcd(c_1, m_1) = 1$.

Since j is fixed and r_1 is constant, ϕ is well-defined.

To show that ϕ is one-to-one:

Suppose that $\phi(\upsilon_1) = \phi(\upsilon_2)$ for $\upsilon_1, \upsilon_2 \in V(G')$. Then

$$\begin{aligned} \phi(\upsilon_1) &= \phi(\upsilon_2) \\ \upsilon_1 c_1 r_i + j &\equiv \upsilon_2 c_1 r_i + j \pmod{n} \\ \upsilon_1 c_1 r_i &\equiv \upsilon_2 c_1 r_i \pmod{n} \\ \upsilon_1 c_1 &\equiv \upsilon_2 c_1 \pmod{m_1} \\ \upsilon_1 &\equiv \upsilon_2 \pmod{m_1} \text{ since } gcd(m_1, c_1) = 1. \end{aligned}$$

Since $0 \le \upsilon_1, \upsilon_2 < m_1$, it follows that $\upsilon_1 = \upsilon_2$. Hence, ϕ is one-to-one.

To show that ϕ is onto: Since any $w \in V(\Gamma(S(i, j)))$, $w = \upsilon s_1 + j$.

Thus, we take $\upsilon \in V(G')$.

To show that is ϕ edge-preserving:

Let $\upsilon_1, \upsilon_2 \in V(G')$.

$$\begin{aligned} \upsilon_1\upsilon_2 \in V(G') &\Longleftrightarrow \upsilon_2 \equiv \upsilon_1 + s_k' \pmod{m_1} \\ &\Longleftrightarrow \upsilon_2 \equiv \upsilon_1 + s_k' \pmod{m_1} \\ &\Longleftrightarrow \upsilon_2 \equiv \upsilon_1 + s_k' + xm_1 \text{ for some integer } x \end{aligned}$$

Thus,

$$
\begin{aligned}
\phi(\upsilon_2) &= \upsilon_2 r_1 + j \\
&= (\upsilon_1 + s_k' + x m_1) + j \\
&= \upsilon_1 s_1 + s_k' + x m_1 r_1 + j \\
&= (\upsilon_1 r_1 + j) + r_k + x m_1 r_1 \\
&= \phi(\upsilon_1) + r_k + x m \\
&\equiv \phi(\upsilon_1) + r_k \pmod{n}
\end{aligned}
$$

Since r_k is a jump size of G and since $\phi(\upsilon_1)$, $\phi(\upsilon_2) \in V(\Gamma(S(i, j)))$, it follows that $\phi(\upsilon_1)\phi(\upsilon_2) \in V(\Gamma(S(i, j)))$.

Suppose that $\phi(\upsilon_1)\phi(\upsilon_2) \in E(\Gamma(S(i, j)))$.

Then

$$
\begin{aligned}
\phi(\upsilon_2) &\equiv \phi(\upsilon_1) + r_k \pmod{n} \\
\upsilon_2 r_1 + j &\equiv (\upsilon_1 r_1 + j) + r_k \pmod{n} \\
\upsilon_2 r_1 &\equiv \upsilon_1 r_1 + r_k \pmod{m_1 r_1} \\
\upsilon_2 &\equiv \upsilon_1 + r_k' \pmod{m_1}
\end{aligned}
$$

Since $r_k' r_1 = r_k$, it follows that r_k' is a jump size of G'. Now, $\upsilon_1, \upsilon_2 \in V(G')$. Hence, $\upsilon_1\upsilon_2 \in V(G')$.

Hence, ϕ is edge-preserving.

This implies that $G' \cong (\Gamma(S(j))$.

Since $S(j)$ for $j = 0, 1, 2, \ldots, r_1 - 1$ partitions $V(G)$ and since $G' \cong (\Gamma(S(j))$, it follows that G is decomposed into r_1 isomorphic circulant subgraphs.

Thefore, $G(n; u, s_1, s_2, \ldots, s_t)$ is a hierarchical circulant network

Let S be the set of all jump sizes of the circulant network G(up to magnitude). Suppose that the order of G is n. We denote the circulant network G by $G(n; S)$.

Theorem 5 *Let $G(n; S)$ be a circulant network.*

Suppose that $s_{i_1}, s_{i_2}, \ldots, s_{i_t} \in S$, $gcd(n; s_{i_j}) = r_{i_j}$ and $r_{i_j} | r_{i_1}$, for $j = 1, 2, \ldots, t$. Then $G(n; S)$ is hierarchical.

Proof. *Let $m_{i_1} r_{i_1} = n$.* Consider the circulant network $G(m_{i_1}; r_1, r_2, \ldots, r_t)$ *where $r_i r_{i_1} = r_{i_j}$ for $j = 1, 2, \ldots, t$.*

Let $S(l) = l, s_{i_1} + l, 2s_{i_1} + l, \ldots, (m_{i_1} - 1) s_{i_1} + l\}$ for $l = 0, 1, \ldots, s_{i_1} - 1$.

Again, for fixed l, the elements of $S(l)$ are distinct and $S(l)$ partitions $V(G(n; S)$ for $l = 0, 1, \ldots, s_{i_1} - 1$.

Consider the induced subgraph $\Gamma(S(l))$ of $S(l)$. Then two vertices ω_1, ω_2, are adjacent in $\Gamma(S(l)$ if $\omega_2 \cong \omega_1 + s$ (mod n), where s is a jump size in $G(n; S)$.

We shall show that $G(m_{i1}; r_1, r_2, \ldots, r_t) \cong \Gamma(S(l))$.

Fr fixed l, define a function ϕ: $V(G') \to V(\Gamma(S(l)))$ by

$$\upsilon = \upsilon c_1 r_{i1} + l \tag{5}$$

where $c_1 r_{i1} = s_{i1}$ and $gcd) m_{i1}, c_1) = 1$.

Since l is fixed and s_1 is constant ϕ is well-defined.

To show that ϕ is one-to-one, onto and edge-preserving:

Actually, the solution is similar to the previous theorem.

Hence, $G(m_{i1}; r_1, r_2, \ldots, r_t) \cong \Gamma(S(l))$.

Since $S(l)$ for $l = 0, 1, 2, \ldots, r_{i1} - 1$ partitions $V(G)$ and since $G' \cong \Gamma(S(j))$, it follows that $G(n; S)$ is composed into r_{i1} isomorphic circulant subgraphs.

Therefore, $G(n; S)$ is hierarchical.

The theorems we have proved do not exchaust all the possible conditions for circulant networks to be hierarchical. It is recommended that future research be conducted to complete the search for all the possible conditions when a circulant network is hierarchical.

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